b. On each of the Geoboards below, make a line perpendicular to the given line. Develop, describe, and justify a rule or procedure for making sure that the two lines are perpendicular.

\[ \text{[Images of Geoboards with lines drawn perpendicular to given lines]} \]

**SECTION 8.2 TWO-DIMENSIONAL FIGURES**

Think of geometric figures that people generally find pleasing, such as those in Figure 8.45. What words would you use to explain why these objects are interesting or appealing? When you look at the various objects and pictures, what similarities do you see between certain objects and shapes—for example, triangles and hexagons?

When we discuss similarities and differences in my class, many geometric terms emerge in the discussion. Some students talk about similar shapes—for example, hexagons in honeycombs and snowflakes, squares in some pictures, and triangles in others. Some students observe that many of the shapes are symmetric. In explaining similarities, some students talk about the angles, the length of sides, or the fact that some figures look similar. Many students observe that even the more complex shapes can be seen as being constructed from simpler shapes, such as triangles and quadrilaterals.

**WHAT DO YOU THINK?**

- In what ways are triangles and quadrilaterals different? In what ways are they alike?
- How are circles and polygons related?
- Can every polygon be broken down into triangles? Why or why not?
- Why do we use two words to name different triangles, but only one word to name different quadrilaterals?

**Figure 8.45**

(a) Carpenter’s Wheel quilt
(b) lamp post
(c) photograph of a snowflake
In a few moments, you will begin a systematic exploration of geometric shapes. Before you do, let us examine a very important framework for understanding the development of children’s geometric thinking. This model was developed by Pierre and Dina van Hiele-Geldorf in the late 1950s and is widely used today. Essentially, the van Hieles found that there are levels, or stages, in the development of a person’s understanding of geometry.2

**Level 1: Reasoning by resemblance**

At this level, the person’s descriptions of, and reasoning about, shapes is guided by the overall appearance of a shape and by everyday, nonmathematical language. For example, “This is a square because it looks like one.” Students at this level may be made aware of the various properties of geometric objects (for example, that a square has four equal sides), but such awareness can be overridden by other factors. For example, if we turn a square on its side, the student may insist that it is no longer a square but now is a diamond.

**Level 2: Reasoning by attributes**

At this level, the person can go beyond mere appearance and recognize and describe shapes by their attributes. A student at this level, seeing the figure above, can easily classify it as a quadrilateral because it has four sides. However, a student at this level does not regularly look at relationships between figures. A student who argues that a figure “is not a rectangle because it is a square” is reasoning at this level.

**Level 3: Reasoning by properties**

At this level, the student sees the many attributes of shapes and the relationships between and among shapes. A student at this level can see that the square and rhombus have many properties in common, such as opposite sides parallel, all four sides congruent, and diagonals that bisect each other and are perpendicular. This enables the student to understand that a square is simply a rhombus with one additional property—all the angles are right angles.

**Level 4: Formal reasoning**

Students at this level can understand and appreciate the need to be more systematic in their thinking. When solving a problem or justifying their reasoning, they are able to focus on mathematical structures.

The investigations in the text and the accompanying explorations have been designed to be consistent with this approach. Accordingly, as you are working in this chapter, reflect on your own thinking. Are you looking at the problem only on a vague, general level? Are you fixated on just one attribute? Are you seeing relationships among triangles, among quadrilaterals? As you move from what to why, are you able to move from solving a problem by random trial and error to being more systematic and careful in your approach?

With this model in mind, let us begin our exploration of shapes with an investigation my students have found to be both fun and powerful. Even more

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2My writing of this section has been informed by Thomas Fox’s “Implications of Research on Children’s Understanding of Geometry” in the May 2000 issue of *Teaching Children Mathematics*, pp. 572–576.
important than knowing the names of all these different shapes will be knowing their properties and the relationships between and among the shapes. It is this knowledge that is used by artists, engineers, scientists, and all sorts of other people.

**INVESTIGATION 8.5**

Recreating Shapes from Memory

For this investigation, you will want to have a pencil and an eraser.

**A.** Look at Figure 8.46 for about 1 second. Then close the book and draw Figure 8.46 from memory.

Check the picture again for 1 second. If your drawing was incomplete or inaccurate, change your drawing so that it is accurate. Check the picture again for 1 second. Keep doing this until your drawing is complete and accurate.

Now go back and try to describe your thinking processes as you tried to re-create the figure. From an information processing perspective, your eyes did not simply receive the image from the paper; your knowledge of geometry helped determine how you saw the picture. What did you hear yourself saying to help you remember the picture? Then read on . . .

**DISCUSSION**

Some students see a diamond and 4 right triangles. Other students see a large square in which the midpoints of the sides have been connected to make a new square inside the first square. Yet other students see four right triangles that have been connected by “flipping” or rotating them.

**B.** Now look at Figure 8.47 for several seconds. Then close the book and try to draw it from memory. As before, check the picture again for a few seconds. If your drawing was incomplete or inaccurate, fix it. Keep doing this until your drawing is complete and accurate. Then go back and describe the thinking processes you engaged in as you tried to re-create the figure . . .

**DISCUSSION**

This figure was more complex. Some students see a whole design and try to remember it.

Some decompose the design into four black triangles and four rectangles as shown in Figure 8.48.

The figure can also be seen as being composed of 9 squares, which can also be seen in Figure 8.48. The four corner squares have been cut to make congruent
right triangles. Each of the other four squares on the border has been cut into two congruent rectangles.

Some students re-created this figure by seeing a whole square and then looking at what was cut out (see Figure 8.49). That is, they saw that they needed to cut out each corner, and they saw that they needed to cut out a rectangle on the middle of each side. Finally, they remembered to cut out a square in the center.

This is a famous quilting pattern called the Churn Dash. When people made their own butter, the cream was poured into a pail called a churn. Then the churner rolled a special pole back and forth with his or her hands. At the end of the pole was a wooden piece called a dash, which was shaped like the figure you saw. If you make many copies of this pattern and put them together, you can see what a Churn Dash quilt looks like.

There are several implications for teaching from this investigation. How a person re-creates the figure is related to the person’s spatial-thinking preferences and abilities. Different people “see” different objects. That is, not everyone sees the figure in the same way. Although there are differences in how people re-create the figure, very few people can re-create the figure without doing some kind of decomposing—that is, without breaking the shape into smaller parts. Being able to do this depends partly on spatial skills and partly on being able to use various geometric ideas (congruent, triangle, square, rectangle) at least at an intuitive level.

Although some people manage to live happy, productive lives at the lowest van Hiele level, an understanding of basic geometric figures and the relationships among them is often helpful in everyday life (for example, in home repair projects and quilting) and in many occupations. Now that your interest in geometric figures has been piqued by this investigation and the pictures at the beginning of the section, let us examine the characteristics and properties of basic geometric shapes.

Before we examine specific kinds of polygons, beginning with triangles, the following investigation will serve to “open your thinking”—to get you to look at polygons not only through the zoom lens, which reveals specific properties and definitions, but also through the wide-angle lens, in which you see all the attributes. For example, a person is not just a person. She might be a mother, a sister, a daughter, a scientist, a Democrat, and so on. Should you become a friend of this person, you come to know her many facets. Similarly, if you become a “friend” of shapes, you come to know the many “sides” of the shapes with which you are working.

**INVESTIGATION 8.6**

Look at the polygons in Figure 8.50. Think of all the attributes, all the characteristics, anything about the polygons that might be important to state or measure. Write down your list before reading on. . . .
DISCUSSION

As you compare the following lists to yours, read actively. If you made the same observation, did you use the same wording? If not, do you understand the wording here? If you missed one of these attributes, why? Do you understand it now? Are there other attributes that you noticed?

<table>
<thead>
<tr>
<th>First figure</th>
<th>Second figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 sides</td>
<td>6 sides</td>
</tr>
<tr>
<td>Top and bottom sides parallel to each other</td>
<td>Top and bottom sides parallel to each other</td>
</tr>
<tr>
<td>Concave</td>
<td>Convex</td>
</tr>
<tr>
<td>2 sets of congruent sides</td>
<td>4 congruent sides; the other pair is also congruent</td>
</tr>
<tr>
<td>2 acute angles, 2 right angles, 2 reflex angles</td>
<td>4 obtuse angles, 2 right angles</td>
</tr>
<tr>
<td>3 pairs of congruent angles</td>
<td>Opposite angles congruent</td>
</tr>
<tr>
<td>1 line of reflection symmetry</td>
<td>No reflection symmetry</td>
</tr>
<tr>
<td>No rotation symmetry</td>
<td>1/2-turn rotation symmetry</td>
</tr>
<tr>
<td>Does not tessellate</td>
<td>Tessellates</td>
</tr>
</tbody>
</table>

A key idea here is to realize that there are lots of attributes and that knowing these attributes and combinations of attributes of a shape helps chemists and physicists to understand the behavior of a molecule or shape; helps builders to know which shapes work together better either in terms of structure and strength or in terms of appearance; and helps artists and designers to make designs that are the most appealing. As you read on, think of multiple attributes and of attributes that different objects have in common.
INVESTIGATION

8.7

Classifying Figures

Before we examine and classify important two-dimensional shapes, we first need to investigate the kinds of possible two-dimensional shapes. Although most of elementary students' exploration of two-dimensional figures will involve polygons and circles, it is important to know that these figures represent just a small subset of the kinds of figures that mathematicians study. Both circles and polygons are curves. A mathematical curve can be thought of as a set of points that you can trace without lifting your pen or pencil. If you watch young children making drawings, you discover that they make all sorts of curves!

As you might expect, if we look at any set of curves, there are many ways to classify them. As I have done throughout this book, rather than giving you the major classifications, I will engage you in some thinking before presenting them. Look at the 13 shapes in Figure 8.51 and classify them into two or more groups so that each group has a common characteristic. Do this in as many different ways as you can, and then read on . . .

**Figure 8.51**

DISCUSSION

One way to sort the figures is shown below. How would you describe the figures in set A and the figures in set B? Do this before reading on . . .

The figures in set A are said to be simple curves. We can describe simple curves in the following way: A figure is a **simple curve** in the plane if we can trace the figure in such a way that we never touch a point more than once. If you look at the figures in set A, you can see that they all have this characteristic; and all the figures in set B have at least one point where the pencil touches twice, no matter how you trace the curve.

*Language*

What other words might you use to describe the intersecting and not intersecting subsets?

Some students use the phrase “trace over,” and others talk about figures that “run over themselves” or “cross themselves.” Other students talk about the set of figures that contain two smaller regions within each figure.
Now look at the curves in sets C and D. How would you describe the figures in set C and the figures in set D? Do this before reading on.

The figures in set C are said to be closed curves. We can describe closed curves in the following way: A figure is a closed curve if we can trace the figure in such a way that our starting point and our ending point are the same. If you look at the figures in set C, you can see that they all have this characteristic; and no matter how you try, you cannot trace the figures in set D with the same starting and ending point.

Now look at the curves in sets E, F, and G. How would you describe the figures in set E, the figures in set F, and the figures in set G? Do this before reading on.

These three sets are interesting for two reasons. First, these sets are likely to be generated in the classroom—both your classroom and the elementary classroom. Second, the language used to describe the three sets poses a challenge, for most people describe the figures in set E as consisting only of curvy lines, the figures in set F as consisting only of straight line segments, and the figure in set G as having both curvy and straight line segments. The challenge here is that when mathematicians use the word curve, this word encompasses both curvy and straight line segments—a curve is a set of points that you can trace without lifting your pen or pencil. There is nothing wrong with students’ use of the terms curvy and straight. What is important is the realization that we are using the words curve and curvy in different ways. We do this all the time in everyday English. Recall the various uses of the word hot in Chapter 1: “It sure is a hot day.” “I love Thai food because it is hot.” “This movie is really hot.”

Most of our investigations of curves will focus on simple closed curves. Looking at the descriptions above, try to define the term simple closed curve before reading on.

We will define a simple closed curve in the plane as a curve that we can trace without going over any point more than once while beginning and ending at the same point. The set of polygons is one small subset of the set of simple closed curves.

At this point, you might want to do the following activity with another student.

- Draw a simple closed curve.
- Draw a simple open curve.
• Draw a nonsimple closed curve.
• Draw a nonsimple open curve.

Exchange figures with another student. Do you both agree that each of the other’s drawings matches the description? If so, move on. If not, take some time to discuss your differences.

There is an important mathematical theorem known as the Jordan curve theorem, after Camille Jordan: Any simple closed curve partitions the plane into three disjoint regions: the curve itself, the interior of the curve, and the exterior of the curve. See the examples in Figure 8.52.

![Figure 8.52](image)

I know that many students’ reaction to this theorem is “Why do we need to prove something that is so obvious?” As mentioned before, being critical (“to examine closely”) is an attitude that I invite. In the examples in Figure 8.52, deciding whether a point is inside or outside is easy. However, look at Figure 8.53. Although this figure is a simple closed curve, it is a rather complicated figure, and such complicated shapes are encountered in some fields of science. Is point A inside or outside? How would you determine this? Think before reading on. . . .

![Figure 8.53](image)

Some wag once remarked that mathematicians are among the laziest people on earth because they are always looking for shortcuts and simpler ways to solve problems. Thus you may be wondering whether someone has
found an easier way to solve these problems. Look at Figure 8.53 to illustrate the method. Start at a point that is clearly outside the shape and draw a line segment connecting that point to the point you are looking at; it helps if you pick an outside point so that the line segment will cross the curve in as few points as possible. Each time you cross a point, it’s like a gate—if you were outside, you are now inside; if you were inside, you are now outside. Thus it is a relatively simple matter to determine that point $A$ is inside the curve.

Now go back to the first shape to see whether point $A$ is inside or outside. As an active reader you will want to work on the figure to verify to your satisfaction that the method just described really does work.

**Polygons**

We are now ready to begin our exploration of **polygons**, which can be defined as simple closed curves in the plane composed only of line segments. Thus the simple closed curve in Figure 8.52(a) is not a polygon, whereas the simple closed curve in Figure 8.52(b) (which looks like the state of Nevada) is a polygon. On any polygon, the point at which two sides meet is called a **vertex**, the plural of which is **vertices**. The line segments that make up the polygon are called **sides**.

The word *polygon* has Greek origins: *poly-* meaning “many,” and *-gon*, meaning “sides.” You are already familiar with many kinds of polygons. Just as we found in Chapter 2 that the names we give numbers have an interesting history, so do the names we give to polygons. The most basic naming classification involves the number of sides (see Table 8.2).

Now that we have a good general definition of the term *polygon*, we can spend time examining triangles, quadrilaterals, and a few specific kinds of polygons.

<table>
<thead>
<tr>
<th>Number of sides</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 sides</td>
<td>Triangle</td>
</tr>
<tr>
<td>4 sides</td>
<td>Quadrilateral</td>
</tr>
<tr>
<td>5 sides</td>
<td>Pentagon</td>
</tr>
<tr>
<td>6 sides</td>
<td>Hexagon</td>
</tr>
<tr>
<td>7 sides</td>
<td>Heptagon</td>
</tr>
<tr>
<td>8 sides</td>
<td>Octagon</td>
</tr>
<tr>
<td>$n$ sides</td>
<td>$n$-gon</td>
</tr>
</tbody>
</table>

**Triangles**

Triangles are found in every aspect of our lives—in buildings, in art, in science (see Figures 8.54 and 8.55). They are truly “building-block” shapes. Every bicycle I have seen has triangles. Bridges will always contain triangles. If you look at the skeleton of buildings, and the scaffolding around the building, you will always see triangles. Why? Rather than give you the answer, we will use the next investigation to think about this question.
INVESTIGATION

8.8

Why Triangles Are So Important

Cut some strips of paper from a file folder or other stiff material. Punch a hole in the ends and use paper fasteners (improvise if you need to; for instance, you can use paper clips). Make one triangle and one quadrilateral, as shown in Figure 8.56. It need not be an equilateral triangle or a square. What do you see? . . .

DISCUSSION

As you saw, the triangle won’t move—we call it a rigid structure. However, the quadrilateral does move; it is not stable. Make another strip and connect two nonadjacent vertices of your quadrilateral. What happens now? It will remain in the shape. If it is a square, it will remain a square; if it is a parallelogram, it will remain a parallelogram. This is because the addition of that diagonal actually created two triangles, which, as you have found, are rigid structures. The next time you walk about campus and about town, look for triangles. You will suddenly see them everywhere!

As you already know, there are many kinds of triangles. A crucial goal of the next investigation is for your own understanding of triangles to become more powerful. Thus, as always, please do the investigation, rather than just reading through it quickly because “I already know this stuff.”
INVESTIGATION 8.9

Classifying Triangles

You will find nine triangles in Figure 8.57. Copy them and cut them out, and then separate them into two or more subsets so that the members of each subset share a characteristic in common. Can you come up with a name for each subset? What names might children give to different subsets? Do this in as many different ways as you can before reading on.

Figure 8.57

DISCUSSION

STRATEGY 1: Consider sides

One way to classify triangles is by the length of their sides: all three sides having equal length, two sides having equal length, no sides having equal length. There are special names for these three kinds of triangles.

- If all three sides have the same length, then we say that the triangle is **equilateral**.
- If at least two sides have the same length, then we say that the triangle is **isosceles**.
- If all three sides have different lengths—that is, no two sides have the same length—then we say that the triangle is **scalene**.

Which of the triangles in Figure 8.57 are scalene? Which are isosceles? Which are equilateral?

STRATEGY 2: Consider angles

We can also classify triangles by the relative size of the angles—that is, whether they are right, acute, or obtuse angles. This leads to three kinds of triangles: right triangles, obtuse triangles, and acute triangles.

- We define a **right** triangle as a triangle that has one right angle.
- We define an **obtuse** triangle as a triangle that has one obtuse angle.
- We define an **acute** triangle as a triangle that has three acute angles.

Many students see a pattern: A right triangle has one right angle, an obtuse triangle has one obtuse angle, yet an acute triangle has three acute angles. What was the pattern? Why doesn’t it hold? Think before reading on.
The key to this comes from looking at the triangles from a different perspective: Every right triangle has exactly two acute angles, and every obtuse triangle has exactly two acute angles; thus a triangle having more than two acute angles will be a different kind of triangle. This perspective is represented in Table 8.3. Does it help you to understand better the three definitions given above?

<table>
<thead>
<tr>
<th>First angle</th>
<th>Second angle</th>
<th>Third angle</th>
<th>Name of triangle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acute</td>
<td>Acute</td>
<td>Right</td>
<td>Right triangle</td>
</tr>
<tr>
<td>Acute</td>
<td>Acute</td>
<td>Obtuse</td>
<td>Obtuse triangle</td>
</tr>
<tr>
<td>Acute</td>
<td>Acute</td>
<td>Acute</td>
<td>Acute triangle</td>
</tr>
</tbody>
</table>

**STRATEGY 3: Consider angles and sides**

This naming of triangles goes even further. What name would you give to the triangle in Figure 8.58?

This triangle is both a right triangle and an isosceles triangle, and thus it is called a right isosceles triangle or an isosceles right triangle. How many possible combinations are there, using both classification systems? Work on this before reading on...

There are many strategies for answering this question. First of all, we find that there are nine possible combinations (see Figure 8.59). We can use the idea of Cartesian product to determine all nine. That is, if set $S$ represents triangles classified by side, $S = \{\text{Equilateral, Isosceles, Scalene}\}$, and set $A$ represents triangles classified by angle, $A = \{\text{Acute, Right, Obtuse}\}$, then $S \times A$ represents the nine possible combinations.

However, not all nine combinations are possible. For example, any equilateral triangle must also be an acute triangle. (Why is this?) Therefore, “equilateral acute” is a redundant combination. However, it is possible to have scalene triangles that are acute, right, and obtuse. Similarly, we can have isosceles triangles that are acute, right, and obtuse.

Name the two triangles in Figure 8.60. Then read on...

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**Mathematics**

Did you notice the geometric balance in Figure 8.59, which represents the Cartesian product of the two sets of triangles?

**Classroom Connection**

Some students have told me that this investigation is huge for them, because they had never thought about organization and classification with respect to triangles. Triangles just were!
Both are obtuse, isosceles triangles. The orientation on the left is the standard orientation for isosceles triangles. Any time we position a triangle so that one side is horizontal, we call that side the base. As stated at the beginning of this chapter, students often see only one aspect of a triangle; for example, they see the triangle at the left as isosceles but not also obtuse, and they see the triangle at the right as obtuse but not also isosceles.

**CLASSROOM CONNECTION**

Children’s development of triangles is fascinating. In several studies, children were given many shapes and asked to identify the shapes. Young children tend to identify the equilateral triangle in the “standard” position (base parallel to the bottom of the page) as a “true” triangle. They will often reject triangles such as the ones below because they are too pointy or turned upside down. Recall level 1 in the van Hiele model. One of my all-time favorite examples occurred when a first-grader was given the pattern shown in Figure 8.61 and was asked to continue the pattern. After studying the pattern, she said, “Triangle, triangle, wrong triangle, triangle, triangle, wrong triangle, triangle. . . . The next shape is a right triangle!”

3It is important to note that this child was doing wonderful thinking—she was seeing patterns, and she was looking at attributes, and she was at the beginning of her understanding of triangles.

As mentioned earlier, one of the new NCTM process standards is representation. As I have also mentioned earlier, one definition of understanding has to do with the quantity and quality of connections the learner can make within and between various ideas. In this case, a new representation of the relationship between and among triangles will help to deepen your understanding. More of these problems can be found in Exploration 8.13. These diagrams are also being used more and more in elementary schools because of their tremendous potential to help children see more relationships and connections.

### INVESTIGATION 8.10

**Triangles and Venn Diagrams**

Recall our work with Venn diagrams in Chapter 2. Venn diagrams can help us understand how concepts are and are not related. Let us take two kinds of triangles: right triangles and isosceles triangles. Draw several right triangles and draw several isosceles triangles. Which of these triangles could be placed in both categories? That is, which are right triangles and also

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is isosceles triangles? Make a Venn diagram that illustrates the relationships between the right and isosceles triangles, and place each triangle in the appropriate region of the diagram.

Now draw several acute triangles and draw several equilateral triangles. Which of these triangles could be placed in both categories? How are these triangles related to each other? Make a Venn diagram that illustrates the relationships between the acute and equilateral triangles, and place each triangle in the appropriate region of the diagram.

**DISCUSSION**

When we have two groups of objects and look at how they are related, there are three possible relationships. They may be disjoint—each object is in one or the other group; there may be overlap—some objects are in both groups; or one group may be a subset of the other. Each of these relationships (remember the van Hiele levels) is represented with a different diagram, as shown in Figure 8.62.

Because some triangles are both right and isosceles, those triangles can be placed in the center, showing that they belong to both sets (see Figure 8.63). Because equilateral triangles can contain only acute angles, all equilateral triangles are acute triangles. Hence the equilateral triangles are in a ring that is inside the ring that represents acute triangles.

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**Mathematics**

Interestingly, all right isosceles triangles have the same shape. That is, if you found the ratio of lengths of the sides of two right isosceles triangles, converted this to a percent, placed the larger one on a copy machine, and entered that percent in the reduction button, the smaller triangle would come out. We will explore this notion of similarity in Section 9.3.
Triangle Properties

You probably remember that the sum of the measures of the angles in any triangle is 180 degrees. How could you convince someone who does not know that?

This is a case in which the van Hiele levels are instructive. A level 2 activity would be to have the students measure and add the angles in several triangles. If their measuring was relatively accurate, one or more students would see the pattern and offer a hypothesis that the sum is always 180.

An example of a level 3 activity, which I recommend if you have never done it, is to cut off the three corners of a triangle as in Figure 8.64 and then put the three corners together. What do you see?

An example of a level 4 activity would be a formal proof.

Special Line Segments in Triangles

There are four special line segments that have enjoyed tremendous influence in Euclidean geometry: angle bisector, median, altitude, and perpendicular bisector.

A median is a line segment that connects a vertex to the midpoint of the opposite side.

In triangle SAT (Figure 8.65), TR is a median. Hence SR = RA.

A perpendicular bisector is a line that goes through the midpoint of a side and is perpendicular to that side. In triangle PEN (Figure 8.67), MX is a
perpendicular bisector of side $PN$, because $M$ is the midpoint of $PN$ and $MX$ is perpendicular to $PN$.

![Figure 8.67](image)

An **altitude** is a perpendicular line segment that connects a vertex to the side opposite that vertex. In some cases, as in $\triangle SAT$ in Figure 8.68, we need to extend the opposite side to construct the altitude.

In triangle $ABC$ (Figure 8.68), $BF$ is an altitude. Hence, $m \angle BFA = m \angle BFC = 90^\circ$.

In triangle $STA$, $TP$ is an altitude. Hence, $m \angle TPA = 90^\circ$.

![Figure 8.68](image)

Many students have trouble with the idea of an altitude being outside the triangle. This happens when we have an obtuse triangle oriented with one side of the obtuse angle as the base of the triangle. If you are having trouble connecting the definition of altitude in these situations, I recommend the following: Trace the triangle and cut it out. Stand it up so that $SA$ is on the plane of your desk and $T$ is above that plane. Now draw a line from $T$ that goes "straight down." What do you notice?

If triangle $STA$ were large enough so that you could stand with your head at point $T$, the length of line segment $TP$ would tell you how tall you were!

After reading the preceding discussion, a skeptical reader might ask, "So what? What is the value of these line segments, other than to mathematicians?" One response is that there is practical value, and another response is that there are some really cool relationships here. Let me explain.

The segment with the most practical value is the median. The point where all three medians meet is called the centroid and is the center of gravity of the triangle (see Figure 8.69). That is, if you found this point, made a copy of the triangle with wood, and placed the triangle on a nail at that point, the triangle would balance. Center of gravity is related to balance and is an important concept in the design of many objects—cars, furniture, and art, to mention just a few.

The point where the three perpendicular bisectors meet is called the circumcenter (see Figure 8.70). It turns out that this point is equidistant from each of the three vertices of the triangle. Thus you can draw a circle so that all three vertices of the triangle lie on the circle and the rest of the triangle is inside the circle. We say that the circle circumscribes the triangle.
The point where the three angle bisectors meet is called the incenter (see Figure 8.71). It turns out that this point is equidistant from each of the three sides of the triangle. Thus you can draw a circle so that the circle touches (is tangent to) each of the three sides of the triangle; in this case, the circle is inside the triangle. We say that the circle is inscribed in the triangle.

The point where the three altitudes meet is called the orthocenter (see Figure 8.72). The orthocenter has connections to other geometric ideas. For example, there is a connection between the construction of a parabola and the orthocenter of a triangle.

There are two amazing things about these line segments. The first is that in any triangle there are three medians, three perpendicular bisectors, three angle bisectors, and three altitudes. In each case, the three line segments will always meet at a single point—they are concurrent. That is, the three medians will always meet at one point in any triangle, the three perpendicular bisectors will always meet at one point in any triangle, and so on. Now, except for the case of the equilateral triangle, these points are not the same point. In fact, in a scalene triangle, the four points are all different. However (and this is the other cool thing), there is a relationship among the centroid, circumcenter, and the orthocenter. In any triangle, they are collinear! Do you understand this? Figure 8.73 illustrates this fact, which Leonard Euler first discovered. We even call the line containing these three points an Euler line. If you are curious, you can type in these terms on a search engine. You will even find websites that have interactive features allowing you to see the situations where these four points are distinct but collinear and the situations where they converge into one point.

**Figure 8.71**

**Figure 8.72**

**Figure 8.73**

### Congruence

As you have seen from your geometry explorations, questions sometimes arise about whether two figures are “the same” or not. Such observations and questions deal with the idea of congruence.

At an informal level, we can say that two figures are congruent if and only if they have the same shape and size. An informal test of congruence is to see whether you can superimpose one figure on top of the other. This is closely connected to how children initially encounter the concept and is related to the dictionary definition: “coinciding exactly when superimposed.” That is, if one figure can be superimposed over another so that it fits perfectly, then the two figures are congruent.

Formally, we say that two polygons are congruent if and only if all pairs of corresponding parts are congruent. In other words, in order for us to conclude that two polygons are congruent, two conditions have to be met: (1) Each corresponding pair of angles have the same measure and (2) each corresponding pair of sides have the same length. We use the symbol ≅ to denote congruence.

For example, in Figure 8.74, triangle CAT and triangle DOG are congruent iff $\angle C \cong \angle D$, $\angle A \cong \angle O$, $\angle T \cong \angle G$, $\overline{CA} \cong \overline{DO}$, $\overline{AT} \cong \overline{OG}$, and $\overline{TC} \cong \overline{GD}$.  

The notions of congruent and equal are related concepts. We use the term **congruence** when referring to having the same shape, and we use the term **equal** when referring to having the same numerical value. Thus we do not say that two triangles are equal; we say that they are congruent. Similarly, when we look at line segments and angles of polygons, we speak of congruent line segments and congruent angles. However, when we look at the numerical value of the line segments and angles, we say that the lengths of two line segments are equal and that the measures of two angles are equal.

Congruence is a big idea, both in geometry and beyond the walls of the classroom. Many important properties and relationships come from exploring congruence. The following investigations (and the related explorations) will help move your understanding of congruence to higher van Hiele levels, from the "can fit on top," geometric reasoning by resemblance, to understanding that all corresponding parts are congruent, to geometric reasoning by attributes, to geometric reasoning by properties, which we shall examine now.

**Outside the Classroom**

When do we need congruence in everyday life or in work situations? Take a few minutes to think about this before reading on . . .

Congruence is important in manufacturing; for example, the success of assembly-line production depends on being able to produce parts that are congruent. Henry Ford changed our world by conceiving of making cars not one at a time but as many sets of congruent parts. For example, the left front fender of a 2003 Dodge Caravan is congruent to the left front fender of any other 2003 Dodge Caravan. One of the differences between a decent quilt and an excellent one is being able to ensure that all the squares are congruent. This is quite difficult when using complex designs. Most of the manipulatives teachers use with schoolchildren (Pattern Blocks, unifix cubes, Cuisenaire rods, and fraction bars) have congruent sets of pieces.

**INVESTIGATION 8.11**

**Congruence with Triangles**

For this you need a protractor and a compass, although you can improvise without them. Do both questions on a blank sheet of paper. First, see how many different triangles you can make that have the following attributes: The base is 50 millimeters (mm). The angle coming from the left side of the base is 30°, and the side coming from the left side of the base is 30 mm. Second, see how many different triangles you can make that have the following attributes: The base is 30 mm. The angle coming from the left side of the base is 30°, and the side coming from the right side of the base is 25 mm. Do this before reading on . . .
DISCUSSION

There is only one triangle that can be drawn in the first case. However, in the second case, there are two possible triangles. Thus there is not enough information about the triangle to specify exactly one triangle (see Figure 8.75).

FIGURE 8.75

In Section 8.1, we said that two points determine a line; here, we are looking at what determines a triangle. This notion of determining or specifying is important in mathematics, both for congruence (When are two figures congruent?) and for definitions (How much do we need to specify to determine a shape?). For example, we can define a rectangle as a quadrilateral with four equal angles. Now a rectangle has many more properties than four congruent angles. However, mathematicians have discovered that this information—quadrilateral, four congruent angles—is sufficient so that only rectangles can be drawn that meet that criteria. Thus the notion of “determines” is an important one in mathematics. In elementary school we do not get terribly technical, but that is not the same as saying we just have fun and play around. When we ask children to explore well-focused questions, their understanding of shapes and relationships between and among shapes and their ability to see and apply properties can grow tremendously. When this happens, high school mathematics makes much more sense!

Quadrilaterals

We found that we could describe different kinds of triangles by looking at their angles or by looking at relationships among their sides. With quadrilaterals, which have one more side, new possibilities for categorization emerge: parallel sides, adjacent vs. opposite sides, relationships between diagonals, and the notion of concave and convex. Thus, how we go about naming and classifying quadrilaterals is not the same as how we name and classify triangles.

In this book, we will define the following kinds of quadrilaterals:

- A trapezoid (Figure 8.76) is defined as a quadrilateral with at least one pair of parallel sides.

- A parallelogram (Figure 8.77) is defined as a quadrilateral in which both pairs of opposite sides are parallel.

- A kite (Figure 8.78) is defined as a quadrilateral in which two pairs of adjacent sides are congruent.
• A rhombus (Figure 8.79) is defined as a quadrilateral in which all sides are congruent.

• A rectangle (Figure 8.80) is defined as a quadrilateral in which all angles are congruent.

• A square (Figure 8.81) is defined as a quadrilateral in which all four sides are congruent and all four angles are congruent.

An active reader may have noted that there are many other possible categories of quadrilaterals. For example, there are many different quadrilaterals that have at least one right angle or exactly three congruent sides. Just as we called the triangle with no sides congruent a scalene triangle, we could call a quadrilateral with no sides congruent a scalene quadrilateral. If we have names for quadrilaterals with two pairs of adjacent sides congruent and quadrilaterals with two pair of opposite sides congruent, why stop there? The primary reason why we have names for the quadrilaterals described above is that these are the sets that mathematicians have found interesting, and specifying other groups didn’t lead to anything beyond those groups; it just stopped, like a road that went nowhere. For example, we could define a rightquad as a quadrilateral with at least one right angle, and there are many different-looking quadrilaterals that would be rightquads. However, there is no name for this set of quadrilaterals because examining them as a class wasn’t found to be useful. That is, it didn’t lead to practical applications or to other discoveries.

Diagonals One characteristic of all polygons with more than three sides is that they have diagonals. The more sides in the polygon, the more diagonals. This term is probably familiar to most readers. However, before reading the definition of diagonal below, stop and try to define the term yourself so that it works for all polygons, not just squares and other quadrilaterals. Then read on . . .

A diagonal is a line segment that joins two nonadjacent vertices in a polygon.

Figure 8.82 shows two different diagonals. One of the exercises will ask you to find patterns to determine the number of diagonals in any polygon.

Angles in quadrilaterals We know that the sum of the measures of the angles of any triangle is 180 degrees. Will the sum of the measures of the four angles of any quadrilateral also be equal to one number, or will there be
Section 8.2 / Two-Dimensional Figures

It turns out that for any quadrilateral, the sum of the measures of the four angles is 360 degrees. The following discussion is an informal presentation of one proof. If we draw a generic quadrilateral \( QUAD \) and one diagonal, as in Figure 8.83, what do you notice that might be related to this proof?

If you see two triangles and connect this to your knowledge that the sum of the measures of the angles of a triangle is 180 degrees, you have the key to the proof. That is, you can conclude that the sum of all six angles must be 360 degrees. However, these six angles are equivalent to the four angles of the quadrilateral!

**INVESTIGATION 8.12**

Look at the three quadrilaterals below. Think about their various attributes. Recall Investigation 8.6. Now answer the following question: Which two of the quadrilaterals in Figure 8.84 are most alike and why? Do your thinking and write your response before reading on.

**DISCUSSION**

It is questions like this that help children, and adults, to move up the van Hiele levels. A good case can be made for different answers. Let us begin by simply listing various attributes.

- 1 pair of \( \equiv \) sides
- 1 pair parallel sides
- 2 right angles
- 1 obtuse angle
- 1 acute angle
- 0 reflex angles
- convex

- 2 pairs of \( \equiv \) sides
- 0 parallel sides
- 1 right angle
- 2 obtuse angles
- 1 acute angle
- 0 reflex angles
- convex

- 2 pairs \( \equiv \) of sides
- 0 parallel sides
- 0 right angles
- 0 obtuse angles
- 3 acute angles
- 1 reflex angle
- concave

This list reflects various attributes on which we focused: congruence, parallel, angles, and shape (convex and concave, which will be formally defined soon). On the one hand, the two figures at the right are both kites and therefore “belong” together under that name. On the other hand, the two figures on the left are both convex and both have right angles, so there is much in common between the two of them also.

One of the most puzzling aspects of how mathematics has generally been taught is how much of it is simply learning and reciting facts and theorems that other people have learned. We ask our students to study mathematics, but we rarely let them do mathematics. If we were to teach art this way, students would...
learn techniques and be tested on how well they understood those techniques, but they would never get to do art. Most of the various groups advocating change in how mathematics is learned want to present students with problems where solving the problem means not simply applying an algorithm they have learned but, rather, involves what we call mathematical thinking. I have sought to give the activities in *Explorations* this flavor. It is more difficult to incorporate this “doing mathematics” flavor into the textbook, but the discussion of most investigations illustrates different solution paths to counter the widely held misconception that there is “one best way.” To a greater degree than many of the investigations in the text, the investigation that follows has this flavor of *doing* mathematics.

**INVESTIGATION 8.13**

This investigation brings together the notion of attributes and the notion of determinism (e.g., two points determine a line), which are two of the big ideas of geometric thinking. How many different kinds of quadrilaterals can you make that have exactly two adjacent right angles? Play around with this for a while on a piece of paper. Sketch different quadrilaterals that have exactly two adjacent right angles. What do you see? Can you make any conjectures? Can you prove them? Use whatever tools are available. As you do, try to push yourself beyond random trial and error to being more systematic, to thinking “What would happen if,” to looking at your solutions to see what they have in common.

**DISCUSSION**

Figure 8.85 shows three of many possibilities.

![Figure 8.85](image)

What do they all have in common besides two adjacent right angles? Think before reading on.

They all have two parallel sides. That means they are trapezoids. A curious reader might now be asking whether it is possible not to get a trapezoid. What do you think? How might you proceed, rather than just using random trial and error? Think before reading on.

If you took high school geometry, you might be remembering a theorem that said something like this: If two lines form supplementary interior angles on the same side of a transversal, then the lines are parallel. When we limit our investigation to two adjacent right angles, we make two interior angles that are also supplementary. Thus we know that the opposite sides of this quadrilateral must be parallel. Hence the condition of adjacent right angles determines a trapezoid. Although we can vary the height of the figure and the lengths of the opposite sides, we can get only trapezoids. Figure 8.86 illustrates this.
What if you make a quadrilateral where the two right angles are not adjacent to each other, but opposite each other? Is it possible to have a concave quadrilateral with two right angles? These questions will be left as exercises.

A critical reader might be thinking this is just an exercise for students. Mathematicians don’t do this kind of stuff. But we do! A great deal of mathematics has been developed by mathematicians asking and exploring questions like “I wonder whether this is possible?” and “I wonder what would happen if...” In Chapter 9, I will talk more about a housewife who became intrigued by the question of under what conditions a pentagon would tessellate. Her work was noticed by a mathematician who encouraged her, and the work she did made a significant contribution to the field of tessellations. My belief is that if you experience the doing of mathematics in this course, you will do this with your children, who will then retain the love of numbers and shapes that they almost universally bring to kindergarten and first grade!

We used Venn diagrams to deepen our understanding of triangles in Investigation 8.10. We will do so again with quadrilaterals.

INVESTIGATION 8.14

Relationships Among Quadrilaterals

Consider the Venn diagram and set of quadrilaterals shown in Figure 8.87. What attributes do all of the quadrilaterals in the left ring have? What attributes do all of the quadrilaterals in the right ring have? By the nature of Venn diagrams, the quadrilaterals in the middle section have attributes of both right and left.

**DISCUSSION**

There are a number of ways to answer the question. Let us begin at the most descriptive level.

In the left ring, all of the shapes have four right angles, they all have opposite sides congruent, and they all have opposite sides parallel. Some students state it slightly differently, saying that the shapes all have two pairs of congruent sides and two pairs of parallel sides.

In the right ring, all the shapes also have opposite sides congruent and opposite sides parallel. Another attribute they possess is that all four sides are congruent. The shape that is in the center, by definition, must have the attributes of both rings—four right angles and four congruent sides.

There is a name for figures that are in both rings—squares.
There is a name for figures that are in the left ring—rectangles.
There is a name for figures that are in the right ring—rhombi.
Using set language from Chapter 2, we say that squares are the intersection of rectangles and rhombi. Some students will have noticed that all of these shapes are parallelograms. Thus we could actually add another ring encircling all the shapes in Figure 8.87. That is, all rectangles are parallelograms, all rhombi are parallelograms, and all rhombi are parallelograms.

**Relationships Among Quadrilaterals**

It turns out that we can view the set of quadrilaterals in much the same way we view a family tree showing the various ways in which individuals are related to others. Figure 8.88 shows one of many ways to represent this family tree for quadrilaterals. Take a few moments to think about this diagram and to connect it to what you know about these different kinds of quadrilaterals. Write a brief description. Does it make sense? Does it prompt new discoveries in your mind?

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**Mathematics**

People not familiar with mathematics often express irritation with the mathematical statement that a square is a rectangle. However, it is really not that weird. Consider this question: Are humans primates? Are humans mammals? If you have had any biology, the answer is, "Of course!" Same thing. We can go from general (polygon) to a subset (quadrilateral) to a yet smaller subset (rectangle) to an even smaller subset (square). Similarly, we can go from general (animal) to a subset (mammal) to a smaller subset (primate) to a yet smaller subset (human).

---

**Figure 8.88**

One way to interpret this diagram is to say that any figure contains all of the properties and characteristics of the ones above it. The quadrilateral at the top represents those quadrilaterals that have no equal sides, no equal angles, and no parallel sides; this is analogous to the scalene triangle. The kite and the trapezoid represent two constraints that we can make: two pairs of congruent, adjacent sides or one pair of sides parallel. If we take a kite and require all four sides to be congruent, we have a rhombus. If we take a trapezoid and require both pairs of opposite sides to be parallel, we have a parallelogram. If we require the angles in a parallelogram to be right angles, we have a rectangle. If we require all four sides of a parallelogram to be congruent, we have a rhombus. Both the rhombus and the rectangle can be transformed into squares with one modification—requiring the rhombus to have right angles or the rectangle to have congruent sides. A key point is to begin to see connections and relationships among figures. Many students find, in this course, that their picture of
geometry changes from looking like a list of definitions and properties to looking more like a network with connections among the various figures. This quadrilateral family tree can also help students to realize why mathematics teachers say that a square is a rectangle and it is a rhombus: It has all the properties of each! In everyday language, we say that a square is a special kind of rhombus and a special kind of rectangle. In mathematical language, we say that the set of squares is a subset of the set of rhombuses and a subset of the set of rectangles. The Venn diagram in Figure 8.89 illustrates this relationship.

Convex Polygons

Another concept that emerges with polygons having four or more sides is the idea of convex. Before reading on, look at the two sets of polygons in Figure 8.90, convex and concave (not convex). Try to write a definition for convex. Then read on.

A polygon is convex if and only if the line segment connecting any two points in the polygonal region lies entirely within the region.

If a polygon is not convex, it is called concave or nonconvex.

Looking at diagonals is an easy way to test for concave and convex. If any diagonal lies outside the region, then the polygon is concave. In polygon ABCDE in Figure 8.91, the diagonal AD lies outside the region.

Other Polygons

Although most of the polygons we encounter in everyday life are triangles and quadrilaterals, there are many kinds of polygons with more than four sides. Stop for a moment and think of examples, both natural and human-made objects. Then read on.
All of the figures in Figure 8.92 are polygons.

• The stop sign is an octagon—an eight-sided polygon.
• The common nut has a hexagonal shape—a six-sided polygon.
• The Pentagon in Washington has five sides.

Let us examine a few important aspects of polygons with more than four sides.

First, we distinguish between regular and nonregular polygons. What do you think a regular pentagon or a regular hexagon is? How might we define it? Think about this and write down your thoughts before reading on.

A regular polygon is one in which all sides have the same length and all interior angles have the same measure.

What do we call a regular quadrilateral? What about a regular triangle? Is it possible for a regular polygon to be concave?

Think about these questions before reading on.

A regular quadrilateral is called a square. A regular triangle is called an equilateral triangle. A regular polygon cannot be concave.

A critical reader might be wondering whether you can have a polygon—let's say a pentagon—where all the sides have the same length but not all the angles have the same measure. And what about the converse: Can you have a pentagon where all the angles have the same measure but not all the sides have the same length? What do you think? This will be left as an exercise.
INVESTIGATION 8.15

Sum of the Interior Angles of a Polygon

We found that the sum of the degrees of the interior angles of any triangle is 180, and the sum for any quadrilateral is 360. What do you think is the sum of the interior angles of a pentagon? Can you explain your reasoning? Can you find a pattern in this progression that will enable you to predict the sum of the interior angles of any polygon—for example, one with 10 sides or with 100 sides? Work on this before reading on.

DISCUSSION

From Table 8.4, many students can see that the sum increases by 180 each time but cannot come up with the general case. The solution to this question comes from connecting the problem-solving tool of making a table to the seeing of patterns to seeing “increases by 180” as equivalent to “these are multiples of 180.” We can represent this equivalent representation in a fourth column that contains 180, and so on. What do you see now? Then read on.

The number we multiply 180 by is 2 less than the number of sides in the polygon. Therefore, the sum of the angles of a polygon having \( n \) sides will be equal to \( (n - 2)180 \).

The next step is to ask why. That is, this investigation takes an unproved hypothesis that the sum will increase by 180 degrees each time we add a side. If that hypothesis is true, then it follows that the sum for any polygon will be \( 180(n - 2) \), but why is this true? What do you think? That question will be left as an exercise.

There are two kinds of angles that we will add to our consideration. We have spoken of interior angles; we can also speak of exterior angles. Draw what you think would be the exterior angles of the polygons in Figure 8.93. Then read on.

<table>
<thead>
<tr>
<th>Sides</th>
<th>Sum</th>
<th>Reasoning</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>180</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>360</td>
<td>Increases by 180 each time</td>
</tr>
<tr>
<td>5</td>
<td>540</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>720</td>
<td></td>
</tr>
</tbody>
</table>
CHAPTER 8 / Geometry as Shape

This is one of those cases where mathematics is counterintuitive. An exterior angle is an angle formed by a side of the polygon and the extension of the side adjacent to that side. Now go back and see whether you can draw the exterior angles for the two polygons. How many exterior angles does a polygon have? Then, and only then, look below.

The diagrams in Figure 8.94 illustrate the exterior angles. As you may have guessed, there is an exterior angle associated with each interior angle.

Amazingly, the sum of the exterior angles of any polygon is 360 degrees. You can see the proof for yourself by placing a pencil on vertex C of the triangle above. Now "walk" the pencil around the triangle. That is, move it to vertex O, now rotate the pencil so you are pointing toward W; note that the amount of turn is exactly the exterior angle at O. Now walk the pencil to W. Turn the pencil at W so you are pointing toward C; again, the amount of turn is equal to the exterior angle at W. Finally walk to C and turn to face O. You find that the pencil has made one complete turn. Do the same with quadrilateral PONY. Once again, you find you make one complete turn.

All polygons have interior and exterior angles. When we focus on regular polygons, there is an additional kind of angle. A central angle of a regular polygon consists of a vertex at the center of the polygon and the two sides connecting the center to adjacent vertices of the polygon (see Figure 8.95). The size of the central angle is not difficult to determine if you recall that one complete turn around a point is equal to 360 degrees. The measure of the central angle of a regular \( n \)-gon (a polygon with \( n \) sides) is equal to \( \frac{360}{n} \) degrees. This and other theorems developed in this section are used to advance our understanding of the structure of geometry, which is then used by people in various fields. For example, graphic artists use their sense of geometric structure to make very beautiful and intricate designs, both on everyday objects like fabric and posters and as pieces of art.
Curved Figures

There is one more class of two-dimensional geometric figures that we need to discuss: those figures that are composed of curves that are not line segments.

How many words do you know that describe such shapes? Think and then read on.

There are many such geometric figures—for example, circle, semicircle, spiral, parabola, ellipse, hyperbola, and crescent (see Figure 8.96).

![Figure 8.96](image)

In this course, we will focus on the simplest of all curved geometric figures: the circle. Stop for a moment to think about circles. How would you define a circle? Try to do so before reading on.

A circle is the set of points in a plane that are all the same distance from a given point, the center.

You are probably familiar with the following basic vocabulary for circles.

In Figure 8.97, $C$ is the center of the circle.

The line segment $CA$ is called a radius, the plural of which is radii.

The line segment $AB$ is called a diameter.

The line segment $XY$ is called a chord.

The line $PQ$, which intersects the circle only at point $D$, is called a tangent.

An arc is any part of a circle. We use two letters to denote an arc if the arc is less than half of the circle. However, for larger arcs, we use three letters. Do you see why?

We do this to distinguish between the arc at the left ($A$) in Figure 8.98 and the arc at the right ($ABD$), because both of them have points $A$ and $D$ as endpoints.

![Figure 8.98](image)

On the basis of these descriptions and your previous experience with circles, try to write a definition for each of these terms. Then compare your definitions with the ones below.
A radius of a circle is any line segment that connects a point on the circle to the center.
A diameter of a circle is any line segment that connects two points of the circle and also goes through the center of the circle.
A chord is any line segment that connects two points of the circle. Thus, a diameter is also a chord.
A tangent line intersects a circle at exactly one point.
An arc is a subset of a circle—that is, a part of a circle.

Coordinate Geometry

We have explored some of the territory of two-dimensional geometry. There is so much more! Most of our focus has been on polygons, especially triangles and quadrilaterals. We hope your understanding has moved to higher levels in the van Hiele model. One of the themes of this book is multiplicity—multiple solution paths to most problems, multiple connections between and among ideas, and multiple ways of representing many ideas. One of the most powerful ways to illustrate the value of multiplicity is through coordinate geometry. Up to now, all of our exploration has been without coordinates—that is, we have simply used sketches of geometric figures. Let us look at what is added when we place the figures in a coordinate plane.

A brief review of the Cartesian coordinate system

Any point on a plane can be represented by an ordered pair. The first number represents the point’s horizontal distance from the center of the coordinate plane, which is called the origin and is denoted by the ordered pair (0, 0). The second number represents the point’s vertical distance from the origin. At some time during the development of mathematics, mathematicians adopted the convention that right is positive, left is negative, up is positive, and down is negative.

As we noted at the beginning of this chapter, a powerful contribution of the Greeks was to move us from the how to the why. In this chapter, we have examined some geometric proofs, and you have explored proofs in Explorations. It turns out that some geometric proofs that are very difficult in “normal” representation are actually quite simple with coordinate geometry. Before we get to some proofs, let us do one investigation to review your skills with the coordinate system.

INVESTIGATION

8.16

What Are My Coordinates?

First, let’s play a game. I’m thinking of a rectangle. Three of its coordinates are (3, 5), (7, 5), and (3, 10). What is the fourth coordinate? Do this yourself before reading on. . . .
DISCUSSION

Using their basic understanding of the properties of rectangles and their intuitive, visual understanding of the coordinate plane (see Figure 8.99), most people will deduce that the fourth coordinate is $(7, 10)$.

Now, let us look at how to make this pattern even more obvious. Let us move the parallelogram so that the bottom left vertex is at the origin (see Figure 8.100). That is, we will move the whole rectangle 3 units to the left and 5 units down. In Chapter 9, we will call this a $(-3, -5)$ translation. Now what are the coordinates of the rectangle? Find this before reading on.

With this strategic placement of the coordinates, the assertion that either the $x$ values or the $y$ values are the same is even easier to see.

We need to develop one more idea before we jump more deeply into coordinate geometry, and that is to learn how to find the distance between two points on the coordinate plane. If you took geometry in high school, you probably memorized this formula. However, if you read this carefully, you will find that it doesn’t have to be memorized. Although it is not simple, it is not hard to understand.

INVESTIGATION

8.17

Understanding the Distance Formula

Let us consider two random points on the plane: $Y = (x_1, y_1)$ and $S = (x_2, y_2)$. If we consider the line segment connecting these two points to be the hypotenuse, we can draw the two sides of the triangle (see Figure 8.101). What must be the coordinates of the third vertex, $E$? Remember the discussions in this chapter about determinism—in this case, both the $x$ and $y$ values of the third vertex are determined. Think before reading on.

DISCUSSION

The coordinates of that third vertex are $(x_2, y_1)$. Do you see why? If not, recall that this vertex is the same horizontal distance from the origin as the top vertex; thus it has the same $x$ value. Similarly, it is the same height above the origin as the left vertex; thus it has the same $y$ value.

Now we can apply the Pythagorean formula: $c = \sqrt{a^2 + b^2}$. This formula simply says that in any right triangle, the length of the hypotenuse is equal to the square root of the sum of the squares of the two sides.

If this seems a bit overwhelming, look at Figure 8.102 and connect it to Figure 8.101. That is, in order to find the distance from $Y$ to $S$, we have to find the distance from $Y$ to $E$ and the distance from $E$ to $S$. Then we will square those distances, add them, and take the square root.

But the distance from $Y$ to $E$ is easy because it’s a horizontal line. It is just the difference between the $x$ values—that is, $x_2 - x_1$. Similarly, the distance from $E$ to $S$ is the difference between the $y$ values—that is, $y_2 - y_1$. 
Thus the distance from \( Y \) to \( S \), substituting our distances into the Pythagorean theorem, must be \( \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \).

Phrased in English, if we square the \( x \) distance and the \( y \) distance, add them, and take the square root, we have the distance between any two points.

Now we are ready for one of the proofs. Recall that a parallelogram is defined as a quadrilateral in which opposite sides are parallel. Using only this knowledge, we can prove that the opposite sides of a parallelogram must be congruent.

**INVESTIGATION 8.18**

**The Opposite Sides of a Parallelogram Are Congruent**

Let us begin by sketching a parallelogram on the coordinate plane (see Figure 8.103). Since the opposite sides are parallel and since, from an algebraic perspective, parallel lines have the same slope, we know that the slopes of the opposite sides are equal. Thus, if we place one vertex at the origin and the next vertex at \( (c, 0) \), we have one side lying on the \( x \) axis. Because we know that \( PL \) and \( NA \) must be parallel, we can draw \( NA \) parallel to the \( x \) axis. Thus, if we let \( N \) be represented as the point \( (a, b) \), we know that the \( y \) coordinate of \( A \) must also be \( b \). Do you see why?

**DISCUSSION**

If not, recall that slope is the ratio of rise to run. That is, \( \frac{y_2 - y_1}{x_2 - x_1} \).

We know that the slope of \( PL \) is 0. Thus the slope of \( NA \) must also be zero. If both \( N \) and \( A \) have the same \( y \) coordinate, \( b \), then the slope is \( (b - b)/(d - a) = 0 \).

Because \( PLAN \) is a parallelogram, the other two sides must also be parallel—that is, they must have the same slope. Let us first determine the slope of \( AL \) and \( PN \).

The slope of \( PN \) is \( \frac{b - 0}{a - 0} = \frac{b}{a} \).

The slope of \( AL \) is \( \frac{b - 0}{d - c} = \frac{b}{d - c} \).

What does this tell us about the relationship among \( a, d, \) and \( c \)?

It means that since the two lines have to have the same slope and thus be parallel, \( d - c \) must be equal to \( a \). That is, \( a = d - c \), which is equivalent to \( d = a + c \).

We can now substitute \( (a + c) \) for \( d \) and have the new representation of the coordinates of the points of the parallelogram (see Figure 8.104).
Next we must prove that the lengths of opposite sides are equal. Let’s begin with the easier case, the horizontal sides. The distance from $P$ to $L$ is $c$; that is, it is $c$. The distance from $N$ to $A$ is $(a + c) - a = c$.

What about the distance from $P$ to $N$ and from $L$ to $A$? Using the distance formula from Investigation 8.17, we can find the length of $PN$—that is, the distance between $P$ and $N$.

\[ PN = \sqrt{(a - 0)^2 + (b - 0)^2} = \sqrt{a^2 + b^2} \]

Now, we find the distance from $L$ to $A$.

\[ LA = \sqrt{((a + c) - c)^2 + (b - 0)^2} = \sqrt{a^2 + b^2} \]

Because the two distances are the same, the lengths of the two line segments are congruent. Therefore, we have proved that the opposite sides of a parallelogram are the same length.

**INVESTIGATION 8.19**

**Midpoints of Any Quadrilateral**

Now let us investigate one of my favorite theorems in geometry. I want you to see it first; then we will prove it. On a blank piece of paper, draw a quadrilateral, any quadrilateral. Find the midpoints of each side. I recommend using the metric side of your ruler. If you don’t have a ruler, you can still find the midpoints. Do you see how? . . . Yes, you can fold the paper to find the midpoints of each side. Now make a new quadrilateral by connecting the four midpoints consecutively. What do you see? 

Do another one. This time, make a scalene quadrilateral—no congruent sides, no parallel sides. What you just observed will always happen. That is, in all cases, you will wind up with a parallelogram. Now, let’s prove that.

**DISCUSSION**

Before we can do the proof, we simply need to know how to find the midpoint of a line segment. In this case, the theorem makes intuitive sense, and I will simply present it. The midpoint of the line segment connecting points $(x_1, y_1)$ and $(x_2, y_2)$, shown in Figure 8.105, is \( \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \). That is, it is the average (mean) of the $x$ values and of the $y$ values.

We will prove the theorem for all quadrilaterals. As we saw from the quadrilateral hierarchy, once we prove a theorem for one quadrilateral, it is true for all quadrilaterals below that quadrilateral. Because we are dealing with any
quadrilateral, we cannot assume any properties—congruence or parallel. Thus we will set one vertex at the origin \((0, 0)\) and one vertex on the \(x\) axis \((a, 0)\). Then the other vertices are at arbitrary points \((b, c)\) and \((d, e)\). See Figure 8.106.

**Figure 8.106**

We will label the original quadrilateral \(HORS\) and the quadrilateral formed by the midpoints \(CANT\).

Using the midpoint formulas, we determine the coordinates of the midpoints and then connect them. We now need to show that the opposite sides must be parallel—that is, that they have the same slope.

Let us begin with \(CA\) and \(TN\).

Slope of \(CA = \frac{c}{2} - 0 = \frac{c}{2}\)

A little algebra makes this next step much easier. If we multiply the top and bottom of this expression by 2, we don’t change the value.

Doing this, we have

Slope of \(CA = \frac{c - 0}{(a + b) - a} = \frac{c}{b}\)

Now for \(TN\).

Slope of \(TN = \frac{\frac{e + c}{2} - \frac{e}{2}}{\frac{b + d}{2} - \frac{d}{2}}\)

If we multiply the top and bottom of this expression by 2, we have

Slope of \(TN = \frac{\frac{e + c}{2} - \frac{e}{2}}{\frac{b + d}{2} - \frac{d}{2}} = \frac{c}{b}\)

Thus we have shown that the slope of \(CA = c/b\) and that the slope of \(TN = c/b\). That is, these two line segments are parallel.

By a similar means, we can show that the slope of \(CT\) and the slope of \(AN\) are equal.

Thus we have proved that if you take the midpoints of *any* quadrilateral and connect them in turn, you will always get a parallelogram. To prove that by other means is much more tedious and difficult.

There is much more to coordinate geometry! We will revisit coordinate geometry in Chapter 9, and you will find coordinate geometry in the upper grades of elementary school!
EXERCISES 8.2

1. Write down all the attributes of each of the figures.
   a. 
   b. 
   c. 

2. In each case below, which two figures are most alike? Explain your reasoning.
   a. 
   b. 
   c. 

3. Classroom Connection This question appeared on the Seventh National Assessment of Educational Progress. In what ways are the two figures below alike? In what ways are they different? You can see the “correct” answers at the back this book.

4. Fill in the table below.

<table>
<thead>
<tr>
<th>Shape</th>
<th>Number of diagonals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quadrilateral</td>
<td>2</td>
</tr>
<tr>
<td>Pentagon</td>
<td></td>
</tr>
<tr>
<td>Hexagon</td>
<td></td>
</tr>
<tr>
<td>Heptagon</td>
<td></td>
</tr>
<tr>
<td>Octagon</td>
<td></td>
</tr>
<tr>
<td>n-gon</td>
<td></td>
</tr>
</tbody>
</table>

5. Describe all the geometric shapes you see in the quilt designs below:
   a. 
   b. 

6. Classroom Connection This question was posted in Teaching Children Mathematics. The children’s responses can be found in the April 2004 issue on pp. 403–406: “Tiffany is creating different shapes on her geoboard. . . . [S]he shares her shape with her friend Leticia [who] notices that she can see triangles and rectangles in the design.”
   a. How many different triangles can you find in the design?
   b. How many different rectangles can you find?
   c. What other polygons do you see in the design?
   d. Create your own design. What shapes can you see? What other questions would you ask a friend about your design?
   e. (My extension) What is the polygon with the most number of sides that you can find in this design?

7. Name at least six different polygons that you can see in this shape. Trace and number each shape.

8. Try to make each figure on Geoboard Dot Paper and on Isometric Dot Paper. If you can make the figure, do so and explain why your figure is an example of the speci-
fied triangle or quadrilateral. If you cannot make it, explain why you think it is impossible to make the figure on that type of grid. Your reasoning needs to be based on properties and attributes (see the van Hiele discussion), as opposed to, for example, “It’s a right triangle because it looks like a right triangle.”

a. acute scalene triangle
b. right isosceles triangle
c. obtuse isosceles triangle
d. equilateral triangle
e. trapezoid
f. kite
g. parallelogram
h. rectangle
i. rhombus
j. square
k. square with no sides parallel to the sides of the paper.

9. For each figure below, write “polygon” or “not a polygon.” If it is a polygon, also write “convex” or “concave.”

10. The definition of a regular polygon states that all sides have the same length and all interior angles have the same measure. Why is the second part of the definition necessary? That is, why can’t we just say that a polygon is a regular polygon if all the sides are the same length?

11. How many different quadrilaterals can you make that have at least one pair of adjacent congruent sides? Sketch and label your figures. For example, you can make many different trapezoids, but they are all trapezoids. See the figures below.

12. a. Is this figure a kite? Why or why not?
   b. Is this figure a rectangle? Why or why not?
   c. Is this figure an isosceles triangle? Why or why not?

13. Write directions for making the following figures. Following your directions, the reader should be able to make the same figure.

14. In each case below, determine whether the two figures are congruent only by using your mind. That is, you cannot trace one figure and see whether it can be superimposed on the other figure. Describe your reasoning—that is, how you arrived at your conclusion.

a. Below are two parallelograms made on a Geoboard.

b. Below are two figures made with tangram pieces.

c. Below are two pairs of hexominoes.

15. Describe all quadrilaterals that have these characteristics. If there is more than one, say so.

a. A quadrilateral with opposite sides parallel
b. A quadrilateral with 4 right angles
c. A quadrilateral with all sides equal
d. A quadrilateral in which the diagonals bisect each other
e. A quadrilateral in which the diagonals are congruent
f. A quadrilateral in which adjacent angles are congruent
g. A quadrilateral in which opposite angles are equal
h. A quadrilateral in which no sides are parallel
i. A quadrilateral with 4 congruent sides and 2 pairs of congruent angles
j. A quadrilateral with 4 congruent angles and 2 pairs of congruent sides

16. Draw each of the following or briefly explain why such a figure is impossible.
   a. An isosceles trapezoid
   b. A concave quadrilateral
   c. A curve that is simple and closed but not convex
   d. A nonsimple closed curve
   e. A concave equilateral hexagon
   f. A concave pentagon having three collinear vertices
   g. A pentagon that has 3 right angles and 1 acute angle

17. In each case, explain and justify your answer—that is, indicate why you think there are none, just one, or many.
   a. How many different hexagons can you draw that have all sides equal but not all angles equal?
   b. How many different hexagons can you draw that have exactly 2 right angles?
   c. How many different hexagons can you draw that have exactly 3 right angles?
   d. How many different hexagons can you draw that have exactly 4 right angles?
   e. How many different hexagons can you draw that have exactly 5 right angles?
   f. Can you make a trapezoid with no obtuse angles?
   g. How many different kinds of quadrilaterals can you make that have exactly two opposite right angles?
   h. Can you make a concave quadrilateral with exactly 2 right angles?
   i. Can you make a concave pentagon with exactly 2 right angles?

18. Use a Venn diagram to represent the relationship between:
   a. scalene and obtuse triangles.
   b. equilateral and isosceles triangles.
   c. parallelograms and rectangles.
   d. rectangles, rhombi, and squares.

19. Write in the labels for each set in the problems below. Justify your choice. Add at least one new figure to one of the regions.
   a. 

20. Show that the family tree for quadrilaterals also holds for the characteristics of the diagonals.

21. In this section, we found that we could name some quadrilaterals in terms of their “ancestors.” For example, we could say that a rhombus is a parallelogram with four equal sides. What other quadrilaterals could we describe in terms of their ancestors?

22. Draw a triangle. Find the midpoints of each side. Connect those points. What is the relationship between the new triangle and the original triangle? Prove it.

23. Take a blank piece of paper. Fold it in half and then fold it in half again. Draw a scalene triangle and then cut it out. You should now have four congruent triangles. For each of the following questions, explain your reasoning.
   a. Put these triangles together to make a large triangle.
   b. Is there only one way or are there different ways?
   c. How could you prove that the new triangle is actually a triangle, as opposed to a figure that is almost a triangle?
   d. Does the new large triangle have anything in common with the original triangle?

24. What is the relationship between the number of sides in a polygon and the total number of diagonals in that polygon?

25. Prove that the sum of the interior angles of any polygon is equal to \(180(n - 2)\), where \(n\) represents the number of sides in the polygon.

26. Describe all possible combinations of angles in a quadrilateral (for example, acute, acute, acute, obtuse). Briefly summarize methods you used other than random trial and error.

27. Trace this circle following #28 onto a blank sheet of paper. Describe as many ways as you can for finding the center of the circle. In each case, explain why the method works.

28. Trace the circle onto a blank sheet of paper. After cutting out the circle and finding the center, fold the paper so that the top point of the circle just touches the center of the circle. Fold the circle again so that another point on the circle just touches the center and the two folds meet at a point. The two folds will be congruent. Fold the circle one more time so that your three folds will all be congruent. What kind of triangle has been created inside the circle? Prove that you will get this kind of triangle by making these three folds.
A 3RIT is a figure made from 3 right isosceles triangles. On the left are two examples of RITs, and on the right are two figures that are not RITs.

29. A 3RIT is a figure made from 3 right isosceles triangles. On the left are two examples of RITs, and on the right are two figures that are not RITs.

30. Make as many different hexiamonds as you can. A hexiamond is made by joining six equilateral triangles—when a side is connected, a whole side connects to a whole side. Briefly (in two or three sentences) describe your method(s) for generating as many shapes as possible. Describe any method(s) other than random trial and error; there are many different ways to be systematic. Cut your hexiamonds out and tape them to a separate piece of paper. (There are more than 9 and fewer than 16.)

31. Find the following points on a sheet of graph paper:
   A (3, 5)  B (5, 5)  C (5, –5)  D (–2, –5)
   E (–2, 5)  F (0, 5)  G (0, –3)  H (3, –3)
   Now connect the points in order. What do you see?

32. Find the distance between the pairs of points:
   a. (2, 3) and (5, 7)  b. (4, 7) and (10, 10)

33. Find the midpoints of the line segments connecting these pairs of points:
   a. (0, 4) and (4, 10)  b. (3, 4) and (7, 12)

34. a. Three vertices of a kite are (6, 8), (9, 11) and (12, 8). What are the coordinates of the fourth vertex?
   b. The two vertices that form the base of an isosceles triangle are (–5, 3) and (2, 3). What are the coordinates of the other vertex?
   c. The coordinates of the endpoints of the hypotenuse of a right triangle are (7, 5) and (3, 1). Find the other vertex. There are two possible solutions.
   d. Three vertices of a parallelogram are (0, 0), (4, 0), and (0, 6). Find the fourth vertex. There are three possible solutions.
   e. A rectangle is oriented so that its sides form vertical and horizontal lines. Two coordinates of a rectangle are (–3, 2) and (7, 6). Do you have enough information to determine the other two coordinates? If so, find them. If not, explain why not and describe what information you would need (for example, a third coordinate, the length of one side, the relative location of one of the points).
   f. Make up a problem similar to the ones above and solve it.

35. We can play a variation of a child’s game called “What am I?” that I will call “Where am I?”
   a. I am a square. The intersection of my two diagonals lies at the point (3, 3), and the length of each of my sides is 6. My sides form horizontal and vertical lines. Where am I?
   b. I am an isosceles triangle. The midpoint of my base is the point (7, –2), my base forms a horizontal line, and my vertex is at the point (7, –9). Oh, I almost forgot to tell you. I am upside down. Where am I?
   c. I am a right isosceles triangle. I have an area of 50 square units. The coordinates of the vertex at which the two sides meet is (0, 0) and my sides form horizontal and vertical lines. Do you have enough information to determine the other two coordinates? If so, find them. If not, explain why not and describe what information you would need.
   d. Make up a problem similar to the ones above and solve it.

36. Determine the coordinates of the vertices of tangram pieces if the bottom left-hand corner is at the origin and if the length of each side of the square is 8.

37. Determine the coordinates of the vertices of the following quilt pattern.