WHAT DO YOU THINK?
• What problem-solving tools do you bring to this course?
• In addition to verifying your computation, how can you verify your answer to a problem?

SECTION 1.2 PROBLEM-SOLVING

In the following NCTM summary of the standard on problem-solving, I have added parenthetical comments that are meant to elaborate on the meaning and importance of each statement. As you read the summary, check the extent to which you understand each statement (both the NCTM’s and mine).

Standard 6: Problem Solving
Instructional programs from prekindergarten through grade 12 enable all students to
• build new mathematical knowledge through problem-solving;
  [This is a major reason for a separate Explorations manual. In your explorations, you will encounter the ideas and concepts of the chapter firsthand, as opposed to simply being shown how to do it.]
• solve problems that arise in mathematics and in other contexts;
  [The 1989 Standards emphasized the importance of developing the knowledge to solve multistep, nonroutine problems, something we will elaborate on later in this chapter.]
• apply and adopt a variety of appropriate strategies to solve problems;
  [I will use the metaphor of a toolbox to develop this aspect.]
• monitor and reflect on the process of mathematical problem-solving.
  [Monitoring and reflecting are essential in order to “own” what you learn, as opposed to just “renting” this knowledge.]

NCTM Principles and Standards for School Mathematics:
(Reston, VA: NCTM, 2000, p. 52)

When you say problem-solving to most people, they think of an image something like Figure 1.1. Many of my students tell me that when they came into the course, their primary learning tool was memorization and their primary problem-solving tool was what they called “trial and error.”

But problems need not be a source of dread. If you have done some of the explorations in the Explorations volume, you have already discovered some new tools for solving problems. In this section, we will examine some of the tools that are essential for solving multistep and nonroutine problems. Think of problems beyond the walls of the classroom that require mathematics. Generally, they are not one-step problems (such as simply dividing $a$ by $b$), nor are they usually just like a problem you have solved before. We do our students a disservice if we lead them to believe that problem-solving is simply memorizing formulas and procedures.

Before we do some problems, stop for a moment. What kinds of problem-solving tools do you bring to this course? Many people will find this exercise more productive if they think of actual problems they have had to solve, such as buying a car, saving for college, or deciding how much food and beverages to buy for a party. Stop and write down your thoughts before reading on. If possible, discuss your ideas with another student also.

Our first problem, though silly, is well known because it nicely illustrates a number of important problem-solving strategies.
INVESTIGATION

1.1 Pigs and Chickens

A farmer has a daughter who needs more practice in mathematics. One morning, the farmer looks out in the barnyard and sees a number of pigs and chickens. The farmer says to her daughter, “I count 24 heads and 80 feet. How many pigs and how many chickens are out there?”

Before reading ahead, work on the problem yourself or, better yet, with someone else. Close the book or cover the solution paths while you work on the problem.

When you come up with an answer, compare it to the solution paths below.

DISCUSSION

STRATEGY 1: Use random trial and error

One way to solve the problem might look like what you see in Figure 1.2.

*FIGURE 1.2*

Unfortunately, trial and error has a bad reputation in schools. The words *trial* and *error* do not sound very friendly. However, this strategy is often very appropriate. In fact, many advances in technology have been made by engineers and scientists who were guessing with the help of powerful computers using what-if programs. A what-if program is a logically structured guessing program. Informed trial and error, which I call **guess–check–revise**, is like a systematic what-if program. Random trial and error, which I call **grope-and-hope**, is what the student who wrote the solution in Figure 1.2 was doing. In this case, the student finally got the right answer. In many cases, though, grope-and-hope does not produce an answer, or if it does produce an answer, the student does not have much confidence that it is correct.

STRATEGY 2: Use guess–check–revise (with a table)

One major difference between this strategy and grope-and-hope is that we record our guesses (or hypotheses) in a table and look for patterns in that table. Such a strategy is a powerful new tool for many students because a table often reveals patterns. Look at Table 1.2. A key to “seeing” the patterns is to make a fourth column called “Difference.” Do you see how this column helps? We will explore the notion of how seeing patterns can enhance our problem-solving ability in the next section.
From the table, we observe that if you add 1 pig (and subtract 1 chicken),
you get 2 more feet. Similarly, if you add 2 pigs (and subtract 2 chickens), you
get 4 more feet. Do you see why? Think before reading on. . . .

Because pigs have 2 more feet than chickens, each trade (substitute 1 pig
for 1 chicken) will produce 2 more legs in the total number of feet. This obser-
vation would enable us to solve the problem in the second guess. Do you see
how . . . ? After the first guess, we need 12 more feet to get to the desired 80 feet.
Because each trade gives us 2 more feet, we need to increase the number of pigs
by 6.

It is important to note that the guesses shown in Table 1.2 represent one of
many variations of a guess–check–revise strategy.

<table>
<thead>
<tr>
<th>Table 1.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
</tr>
<tr>
<td>of pigs</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>First guess</td>
</tr>
<tr>
<td>Second guess</td>
</tr>
<tr>
<td>Third guess</td>
</tr>
<tr>
<td>Fourth guess</td>
</tr>
</tbody>
</table>

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you get 2 more feet. Similarly, if you add 2 pigs (and subtract 2 chickens), you
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Because each trade gives us 2 more feet, we need to increase the number of pigs
by 6.

It is important to note that the guesses shown in Table 1.2 represent one of
many variations of a guess–check–revise strategy.

**STRATEGY 3: Make a diagram**

Some people think in words, others in numbers, and still others in pictures.
Sometimes making a diagram can lead to a solution to a problem. I stumbled
across this approach one day as I was walking around the classroom listening
to students work on this problem in small groups. Figure 1.3 shows what one
student had done. How do you think she had solved the problem? Write your
thoughts before reading on. . . .

I asked her how she had solved the problem. She replied that she had made
24 chickens, which gave her 48 feet. Then she kept turning chickens into pigs
(by adding 2 feet each time) until she had 80 feet! I was thrilled because she had
represented the problem visually and had used reasoning instead of grope-
and-hope. She was embarrassed because she felt she had not done it “mathema-
tically.” However, she had engaged in what I call mathematical thinking.

There are two aspects of this strategy that beg to be elaborated. First, it
illustrates the notion of mathematical modeling, a very important aspect of
mathematical problem-solving that will be discussed in more detail later.
Simply stated, most complex mathematical problems are solved first by mak-
ing a model of the problem—the model makes the problem simpler to see and
thus to solve. The model enables us to do things that we could not do in the
original situation. In this case, the model enables us to do what is biologically
impossible but mathematically possible—we turn chickens into pigs until we
get the right answer!

The other aspect of this problem is that, upon reflection, we realize its enor-
mos potential. For example, what if the problem were 82 heads and 192 feet?
No way you say? True, it would be tedious to draw 82 heads and then 2 feet underneath each head. This is exactly the power of mathematical thinking—you don’t have to do all the drawing. Rather, the drawing stimulates the thinking that will lead to a solution. Think about what the diagram tells us, and then see whether you can solve the problem. . . .

If we drew 82 heads and then drew 2 feet below each head, that would tell us how many feet would be used by 82 chickens and how many feet we would still need. Having made the diagram for the simpler problem, we can do that for this problem without a diagram. That is, 82 chickens will use up 164 feet. Because 192 – 164 = 28, we need 28 more feet—that is, 14 more pigs. So the answer is 14 pigs and 68 chickens. Check it out!

Furthermore, this solution connects nicely to an algebraic solution, as we shall see shortly. I have come to believe that for many students, the best way to understand the more abstract mathematical tools is first to solve problems using more concrete tools and then to see the connections between the concrete approach (in this case, a drawing or guess–check–revise) and the abstract approach (in this case, two equations).

**STRATEGY 4: Use algebra**

Because the range of abilities present among students taking this course is generally wide, it is likely that some of you fully understand the following algebraic strategy and some of you do not. Furthermore, many students enter this and other college math courses believing that the algebraic strategy is the right strategy, or at least the best strategy. Let’s look at an algebraic solution and then see how it connects to other strategies and to the goals of this course.

Go back and review strategies 1 and 2. They both involved a total of 24 pigs and chickens. Can you explain in words why this is so? Think about this before reading on. . . .

Most students say something like “Because the total number of animals is 24” or “Well, 24 animals will have 24 heads.” Therefore, if we say that

\[ p = \text{the number of pigs} \]
\[ c = \text{the number of chickens} \]

then the number of pigs plus the number of chickens will be 24. Hence, the first equation is

\[ p + c = 24 \]

Many students have difficulty coming up with the second equation. If this applies to you, look back at how we checked our guesses when using Strategy 2, guess–check–revise: We multiplied the number of pigs by 4 and the number of chickens by 2 and then added those two numbers to see how close that sum was to 80. In other words, we were doing the following:

\[
\text{(The guess for number of pigs)} \times 4 + \text{(the guess for number of chickens)} \times 2 \\
=p \times 4 + c \times 2
\]

More conventionally, this would be written as

\[ 4p + 2c \]

Using guess–check–revise, we had the right answer when this sum was 80. Thus the second equation is

\[ 4p + 2c = 80 \]
If you solve these two equations, using basic algebra, you will discover that $p = 16$ and $c = 8$.

**The algebraic strategy in perspective** Students who do not do well with equations generally understand this algebraic solution better if they see it after they have used guess–check–revise. The reason for this is simple and is grounded in learning theory: *The teacher’s explanation (or another student’s) is far more meaningful if you can connect it to something you already know.* This is such an important learning principle that I want to elaborate. If I show you how to create the equation before you have attempted the problem on your own, my explanation is not likely to stick. In this case, your knowledge is “Teflon knowledge.” The Teflon molecule was specially designed by chemists so that other materials tend not to stick to it. This is a wonderful property for frying pans to have, but not for human brains! To the extent that new ideas and concepts have connections to ideas and knowledge you already have, to that extent you are more likely to retain (that is, to own rather than rent) that new knowledge.

Algebra can be a very effective strategy for many problems. However, I offer the following cautionary notes with respect to the use of algebra in this course. First, the algebraic strategy is not more mathematical than the guess–check–revise strategy; it is simply more abstract. Second, it is better to use guess–check–revise and be confident of the process and of the answer than to use grope-and-hope to come up with an algebraic equation that you hope works out. Furthermore, guess–check–revise is a tool all of your future students should have, regardless of their age, whereas only a small percentage of your future students will have formal algebraic knowledge before grade 8 or 9.

**Problem-Solving and Toolboxes**

An image of problem-solving I would like to develop in your mind is that of a toolbox. Imagine that your car breaks down and is towed to the garage, where a novice mechanic, right out of training school, is the first to look at it. The novice will probably try a few standard procedures: Insert the key to see what happens, check the battery connections, look for a loose wire, and so on. If none of those strategies work, the novice mechanic will be stumped and will have to summon the senior mechanic. The senior mechanic may try the same basic procedures and may solve the problem by interpreting the results. If this does not solve the problem, the mechanic will have to go to two toolboxes. The first toolbox is a physical one. The second toolbox is a mental one.

At the beginning of this course, many students are like the novice mechanic: When they encounter a problem, they have a limited repertoire of strategies. However, as the course develops, their toolbox grows in two ways. First, the number of tools grows. This is like the novice mechanic’s learning to use the garage’s diagnostic equipment. Second, their ability to use each tool also grows. This is analogous to the novice mechanic’s learning how to use a voltmeter more skillfully.

**Looking back** At this point, I want to introduce another process that most successful students have incorporated into their toolbox. They reflect on their work. That is, after solving a problem, they stop and examine their toolbox—what tools worked better and what new tools they used. Take a few minutes now to look at your notes on the strategies we discussed in Investigation 1.1 and then look at “4 Steps for Solving Problems” on the inside front cover of
the Explorations volume. What tools were used in the various solutions of the pigs-and-chickens problem? What made those tools work better? Then read on.

My list includes

1. Draw a diagram—a picture is often worth a thousand words.
2. Guess–check–revise—starting here will help some students to develop the two equations.
3. Make a table—this increases the possibility of seeing more patterns.
4. Look for patterns—extending the table enables certain patterns to become more visible.
5. Develop an equation—a powerful tool, but also surrounded by quicksand for many.

Two final notes about Investigation 1.1:

1. The strategies discussed in Investigation 1.1 do not represent all the different ways in which the pigs-and-chickens problem has been solved. Because of space restrictions, only four were discussed.
2. Many students think that there is only one tool per problem. An important adaptive belief is that there are often several different ways (using different tools) to solve a problem and that the tools are often used in combination instead of separately.

**Polya’s Four Steps**

George Polya developed a framework for problem-solving that breaks down problem-solving into four distinguishable steps. In 1945, he outlined these steps in a now-classic book called *How to Solve It*.

When you approach a problem—a math problem, a writing assignment, even a personal problem—if you think that you have to come up with an answer immediately and that there is only one “right” way to reach that answer, a solution may seem to be beyond your grasp. But if you break the problem down and creatively and mindfully approach each step of the problem, it generally becomes more manageable. Polya suggests that you first need to make sure you understand the problem. Once you understand the problem and know what you need to find out, you devise a plan for solving the problem. Then you monitor your plan; you check frequently to see whether it is productive or is going down a dead-end street. Finally, you look back at your work. This last step involves more than just checking your computation; for example, it includes making sure that your answer makes sense. For each of these four steps, there are specific strategies that we will explore in this chapter and that you will refine throughout this course.

**Owning versus renting**  Instead of just listing Polya’s strategies, we are going to discover them by putting them into action. You will notice that I often ask you to stop, think, and write some notes. I really mean it! I have come to distinguish between those students who own what they learn and those who simply rent what they learn. Many students rent what they have learned just long enough to pass the test. However, within days or weeks of the final exam, it’s gone, just like a video that has been returned to the store. One of the important differences between owners and renters is that those who own the knowledge tend to be active readers.
Think and then read on . . . Throughout the book, I will often pose a question and ask you to “think and then read on . . .” Rather than just look to the next paragraph and see the “answer,” you will learn much more if you immediately cover up the next paragraph or close the book . . . stop . . . think . . . write down your thoughts . . . and then read on. The phrase “think and read on . . .” is there to remind you to read the book actively rather than passively. An active reader stops and thinks about the material just read and asks questions: Does this make sense? Have I had experiences like this? The active reader does the examples with pen or pencil, rather than just reading the author’s description. I recommend that you keep a journal (your instructor may give you specific instructions). Keeping a journal has many benefits. Many people find that they learn more by writing down their responses to “think and read on . . .” rather than just stopping for a moment to reflect. Many students find that reading over their journals every week or at the end of each chapter provides new insights into the mathematics and into their own beliefs and attitudes about mathematics.

Using Polya’s four steps I encourage you to use Polya’s four steps (on the inside front cover of the Explorations volume) in all of the following ways:

1. Use them as a guide when you get stuck.
2. Don’t rent them, buy them. Buying them involves paraphrasing my language and adding new strategies that you and your classmates discover. For example, many of my students have added one whole step to help reduce anxiety: First take a deep breath and remind yourself to slow down.
3. Just as we did in Investigation 1.1, after you have successfully solved a problem, stop for a moment and reflect on the tools you used. As the course progresses, you should find that most problems involve the use of several strategies, and you should find that you use the tools more skillfully. For example, using “Make a diagram” skillfully involves deciding how precise your diagram needs to be, checking to see that the diagram illustrates the relevant given information, seeing whether the diagram can help you to paraphrase the problem or see it from a new perspective, and so on.

Classroom Connection

A colleague of mine was working through a word problem with her class one day and encouraging the students to think about what they were doing. Suddenly one of the students said, “But you don’t need to do all this stuff you are teaching us; you just know the answer.” She was stunned, and the ensuing discussion was informative. It turns out that many students believe that the difference between a student and a teacher is that the teacher just knows the answer or automatically knows how to get the answer. That is, teachers don’t need such strategies as guess–check–revise, make a table, draw a diagram, look for patterns, etc. The truth is that we do! Virtually all of the most brilliant workers—whether they be engineers, scientists, businesspeople, carpenters, researchers, or entrepreneurs—approach complex problems by using the very tools that are being stressed in this text. Furthermore, even the top people in a field often make hypotheses that seem reasonable (to them and to their colleagues) but that turn out not to be true. So when you are working on these problems, please realize that the tools being discussed in this book are used by people in all kinds of situations.
Why Emphasize Problem-Solving?

Although Polya described his problem-solving strategies back in 1945, it was quite some time before they had a significant impact on the way mathematics was taught. One of the reasons is that the desired outcomes of mathematics instruction were defined too narrowly. If the stated goal of mathematics classes was to learn and practice the “right” techniques to answer textbook problems, chances are that students rarely saw how math applied to situations outside the classroom.

Another reason for the limited impact of Polya’s work is that until recently, “problems” were generally defined too narrowly. For example, many of you learned how to do different kinds of problems—mixture problems, distance problems, percent problems, age problems, coin problems—separately but never realized that they have many principles in common. To use language from the NCTM, there has been too great a focus on single-step problems and routine problems. Consider the examples from the National Assessment of Educational Progress shown in Table 1.3.

**TABLE 1.3**

<table>
<thead>
<tr>
<th>Problem</th>
<th>Percent correct</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Here are the ages of six children: 13, 10, 8, 5, 3, 3</td>
<td>Grade 11</td>
</tr>
<tr>
<td>What is the average age of these children?</td>
<td>72</td>
</tr>
<tr>
<td>2. Edith has an average (mean) score of 80 on five tests.</td>
<td>Grade 11</td>
</tr>
<tr>
<td>What score does she need on the next test to raise her average to 81?</td>
<td>24</td>
</tr>
</tbody>
</table>

Mary M. Lindquist, ed., *Results from the Fourth Mathematics Assessment of the National Assessment of Educational Programs* (Reston, VA: NCTM, 1989), pp. 30, 32.

To solve the first problem, one only has to remember the procedure for finding an average and then use it:

\[
\frac{13 + 10 + 8 + 5 + 3 + 3}{6}
\]

However, there is no simple formula for solving the second problem. Try to solve it on your own and then read on...

To solve this one, you have to have a better understanding of what an average means. One approach is to see that if her average for 5 tests is 80, then her total score for the 5 tests is 400. If her average for the 6 tests is to be 81, then her total score for the 6 tests must be 486 (that is, 81 \times 6). Because she had a total of 400 points after 5 tests and she needs a total of 486 points after 6 tests, she needs to get an 86 on the sixth test to raise her overall average to 81.

Many students still consider the second question to be a “trick” question unless the teacher has explicitly taught them how to solve that kind of problem. However, many employers note that problems that occur in work situations are rarely just like the ones in the book. What employers desperately need is more people who can solve the “trick” problems, because, as one wag put it, “life is a trick problem!”

*The difference between traditional word problems and many real-life problems*  Table 1.4 lists differences between the word problems generally found in textbooks and real-life problems.
When students undertake what some authors call more authentic problems, they come to realize that mathematics is more than just memorizing and using formulas, and they come to value their own thinking.

With respect to problem-solving, the NCTM has urged a turnabout. Traditionally, the teacher “taught” a new concept or skill, and then students did problem-solving. The NCTM has rejected this separation of teaching and problem-solving. That is, problem-solving involves more than just doing word problems to apply the concepts. In this book, we will investigate a wide variety of problems. By solving these problems, you will discover the meaning of the concepts and how to apply them and come to a much deeper understanding of and appreciation for the formulas or procedures you learn.

**INVESTIGATION 1.2**

**How Much Will the Patio Cost?**

Let’s say you are building a patio in your back yard. You have decided to make the patio 12 feet by 8 feet. The local lumber store sells premade patio blocks, which measure 18 inches by 12 inches, for 75¢ each. How much will the patio cost?

Solve this problem on your own, taking time to understand the problem and consider how you might plan to solve it. Then compare your solution and strategies to the ones discussed below.

**DISCUSSION**

**STRATEGY 1: Make a diagram**

Let us examine two kinds of diagrams that students often come up with. Because the blocks are 1 1/2 feet long and 1 foot wide, some students find a piece of

**TABLE 1.4**

<table>
<thead>
<tr>
<th>Textbook word problems</th>
<th>Real-life problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. The problem is given.</td>
<td>1. Often, you have to figure out what the problem really is.</td>
</tr>
<tr>
<td>2. All the information you need to solve the problem is given.</td>
<td>2. You have to determine the information needed to solve the problem.</td>
</tr>
<tr>
<td>3. There is always enough information to solve the problem.</td>
<td>3. Sometimes you will find that there is not enough information to solve the problem.</td>
</tr>
<tr>
<td>4. There is no extraneous information.</td>
<td>4. Sometimes there is too much information, and you have to decide what information you need and what you don’t.</td>
</tr>
<tr>
<td>5. The answer is in the back of the book, or the teacher tells you whether your answer is correct.</td>
<td>5. You, or your team, decides whether your answer is valid. Your job may depend on how well you can “check” your answer.</td>
</tr>
<tr>
<td>6. There is usually a right or best way to solve the problem.</td>
<td>6. There are usually many different ways to solve the problem.</td>
</tr>
</tbody>
</table>
FIGURE 1.4  FIGURE 1.5

graph paper and let each square represent $1/2$ foot, as shown in Figure 1.4. If you make one X on the grid for each patio block, you can determine the total number. What do you get?

Other students simply sketch the problem on a blank piece of paper as shown in Figure 1.5.

In either case, we can see that we will need 64 blocks, and 64 blocks times 75¢ per block is 4800¢ or $48.

**STRATEGY 2: Divide**

There is another way to solve the problem that is quicker but also requires more thinking at the beginning of the problem. If we divide the total area of the patio by the area of one block, the quotient will tell us how many blocks we need.

The total area of the patio is $12 \text{ feet} \times 8 \text{ feet} = 96 \text{ square feet}$.

The area of each block is $1.5 \text{ feet} \times 1 \text{ foot} = 1.5 \text{ square feet}$.

When we divide 96 square feet by 1.5 square feet, we get 64 blocks.

Some readers may be kicking themselves for not thinking of this strategy. In Chapter 3, you will find that one of the meanings of division is repeated subtraction. From one perspective, this problem is asking how many times we can subtract 1.5 from 96.

**STRATEGY 3: Use dimensional analysis**

We can treat the “labels” as algebraic terms. Just as numbers can “cancel,” so too can labels. We will investigate canceling further in Chapter 5. Using dimensional analysis, we have

\[
\frac{96 \text{ square feet}}{1 \text{ block}} = 64 \text{ blocks}
\]

We can treat this as a division-of-fractions problem; thus we need to invert and multiply.

\[
= \frac{96 \text{ square feet}}{1.5 \text{ square feet}} \times \frac{1 \text{ block}}{1 \text{ block}}
\]

Because square feet “cancel,” the meaning of 64 is 64 blocks.