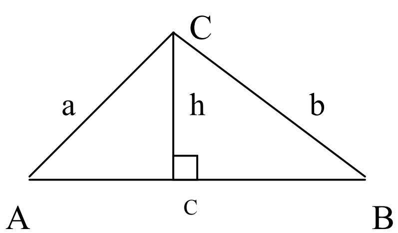
Heron's Formula

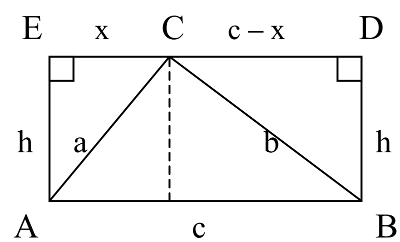
Theorem 7.2.1 (Heron's Formula): If the three sides of a triangle have lengths a, b, and c, then the area of the triangle is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

Where s represents the semi perimeter of the triangle, s = $\frac{1}{2}(a+b+c)$.

Proof: Consider ΔABC, with sides of lengths a, b, and c. Note that $\angle C$ need *not* be a right angle. In the figure to the right, we inscribe ΔABC in rectangle ABDE as shown.





1. Because A$_{rect}$ = $c \cdot h$, A$_{\Delta}$ $= \frac{1}{2}ch$.

2. Using the Pythagorean Theorem, we have

   $h^2 + x^2 = a^2$ and $h^2 + (c-x)^2 = b^2$, so that

   $h^2 = a^2 - x^2$ and $h^2 = b^2 - (c-x)^2$

   By substitution (each expression above equals h^2), we have

   $$a^2 - x^2 = b^2 - (c-x)^2$$

   Then
   $$a^2 - b^2 = x^2 - (c-x)^2$$
   $$a^2 - b^2 = x^2 - c^2 + 2cx - x^2$$
   $$a^2 - b^2 = 2cx - c^2$$

Then $2cx = a^2 - b^2 + c^2$, *so* $x = \dfrac{a^2 - b^2 + c^2}{2c}$

3. Using the result in step 2 with the earlier fact that $h^2 = a^2 - x^2$, we have

$$h^2 = a^2 - \left(\frac{a^2 - b^2 + c^2}{2c}\right)^2$$

Because the right side of the equation is a *difference of two squares* $[A^2 - B^2 = (A+B)(A-B)]$,

$$h^2 = \left(a + \frac{a^2 - b^2 + c^2}{2c}\right)\left(a - \frac{a^2 - b^2 + c^2}{2c}\right)$$

$$h^2 = \left(\frac{2ac}{2c} + \frac{a^2 - b^2 + c^2}{2c}\right)\left(\frac{2ac}{2c} - \frac{a^2 - b^2 + c^2}{2c}\right)$$

$$h^2 = \left(\frac{(a^2 + 2ac + c^2) - b^2}{2c}\right)\left(\frac{b^2 - (a^2 - 2ac + c^2)}{2c}\right)$$

$$h^2 = \left(\frac{(a+c)^2 - b^2}{2c}\right)\left(\frac{b^2 - (a-c)^2}{2c}\right)$$

Factoring the differences of squares,

$$h^2 = \left(\frac{(a+c+b)(a+c-b)}{2c}\right)\left(\frac{(b+a-c)(b-a+c)}{2c}\right)$$

$$h^2 = \frac{(a+c+b)(a+c-b)\,(a+b-c)(b-a+c)}{4c^2}$$

Then $$h = \sqrt{\frac{(a+c+b)(a+c-b)\,(a+b-c)(b-a+c)}{4c^2}} \quad or$$

$$h = \frac{\sqrt{(a+c+b)(a+c-b)\,(a+b-c)(b-a+c)}}{\sqrt{4c^2}}$$

$$h = \frac{\sqrt{(a+c+b)(a+c-b)\,(a+b-c)(b-a+c)}}{2c}$$

$$h = \frac{1}{2c}\sqrt{(a+c+b)(a+c-b)\,(a+b-c)(b-a+c)}$$

To introduce the semiperimeter $s = \frac{1}{2}(a+b+c)$, we modify the expression for h as follows:

$$h = \frac{1}{2c}\sqrt{(a+b+c)(a+b+c-2b)\,(a+b+c-2c)(a+b+c-2a)}$$

$$h = \frac{1}{2c}\sqrt{\frac{2(a+b+c)}{2}\cdot\frac{2(a+b+c-2b)}{2}\cdot\frac{2(a+b+c-2c)}{2}\cdot\frac{2(a+b+c-2a)}{2}}$$

$$h = \frac{1}{2c}\sqrt{\frac{2(a+b+c)}{2}\cdot 2(\frac{a+b+c}{2}-\frac{2b}{2})\cdot 2(\frac{a+b+c}{2}-\frac{2c}{2})\cdot 2(\frac{a+b+c}{2}-\frac{2a}{2})}$$

$$h = \frac{1}{2c}\sqrt{2\frac{a+b+c}{2}\cdot 2(\frac{a+b+c}{2}-b)\cdot 2(\frac{a+b+c}{2}-c)\cdot 2(\frac{a+b+c}{2}-a)}$$

$$h = \frac{1}{2c}\sqrt{16s(s-b)\cdot(s-c)\cdot(s-a)} = \frac{4}{2c}\sqrt{s(s-a)\cdot(s-b)\cdot(s-c)}, \quad so$$

$$h = \frac{2}{c}\sqrt{s(s-b)\cdot(s-c)\cdot(s-a)}$$

4. Recalling from step 1 that $A_\Delta = \frac{1}{2}ch$, we have

$$A_\Delta = \frac{1}{2}c\cdot\frac{2}{c}\sqrt{s(s-a)\cdot(s-b)\cdot(s-c)} \text{ or}$$

$$A_\Delta = \sqrt{s(s-a)\cdot(s-b)\cdot(s-c)}$$